POD 12

$\underset{\sim}{u} \cdot \underset{\sim}{\nabla} T$

$$u_z = U\, 2\left(1 - \frac{r^2}{a^2}\right)$$

$$\rho \hat{C}_p u_z \frac{\partial T}{\partial z} = k \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right)$$

$$T\Big|_{z=0} = T_0 \qquad T\Big|_{r=a} = T_W \qquad \frac{\partial T}{\partial r}\Big|_{r=0} = 0$$

$$T^* = \frac{T - T_0}{T_W - T_0} \qquad r^* = r/a \qquad z^* = \frac{z}{z_c}$$

$$\frac{\rho \hat{C}_p U}{z_c} \Delta T \; 2\left(1 - r^{*2}\right)\frac{\partial T^*}{\partial z^*} = \frac{k \Delta T}{a^2} \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial T^*}{\partial r^*}\right)$$

$$\boxed{\frac{a^2 U}{\alpha z_c}} \; \underbrace{2\left(1 - r^{*2}\right)\frac{\partial T^*}{\partial z^*} = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial T^*}{\partial r^*}\right)}$$

$$" 1 \quad \therefore z_c = U \frac{a^2}{\alpha}$$

$$T^*\Big|_{z^*=0} = 0 \qquad T^*\Big|_{r^*=1} = 1 \qquad \frac{\partial T^*}{\partial r^*}\Big|_{r^*=0} = 0$$

$$T^*_\infty = 1$$

$$q_r\Big|_{r=a} = -k\frac{\partial T}{\partial r}\Big|_{r=a} = -\frac{k\,\Delta T_c}{a}\frac{\partial T^*}{\partial r^*}\Big|_{r^*=1}$$

$$h = \frac{q_r|_{r=a}}{T_w - T_b} \qquad T_b = \frac{\int_0^a T\,U\,2\left(1-\frac{r^2}{a^2}\right)2\pi r\,dr}{\int_0^a U\,2\left(1-r^2/a^2\right)2\pi r\,dr}$$

$$T_b^* = \int_0^1 T^*\,2(1-r^{*2})\,2r^*\,dr^*$$

$$h = \frac{k}{a}\,\frac{\frac{\partial T^*}{\partial r^*}\Big|_{r^*=1}}{(1-T_b^*)}$$

$$T^* = T_\infty^* + T_d^* = 1 + T_d^*$$

$$T_d^*\Big|_{z^*=0} = -1 \qquad T_d^*\Big|_{r^*=1} = 0 \qquad \frac{\partial T_d^*}{\partial r^*}\Big|_{r^*=0} = 0$$

$$2(1-r^{*2})\frac{\partial T_d^*}{\partial z^*} = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial T_d^*}{\partial r^*}\right)$$

$$T_d^* = G(z^*)F(r^*)$$

$$2(1-r^{*2})G'F = G\frac{1}{r^*}(r^*F')'$$

$$\frac{G'}{G} = \frac{(r^*F')'}{2r^*(1-r^{*2})F} = -\sigma^2 \equiv -\lambda$$

$$G = e^{-\lambda z^*}$$

$w(x)$

$$(r^*F')' + \lambda\, 2r^*(1-r^{*2})F = 0$$

$p(x)$

$$F'(0) = 0 \qquad F(1) = 0$$

$$T_d^* = \sum_{n=1}^{\infty} A_n e^{-\lambda_n z^*} F_n(r^*)$$

$$h \cong \frac{k}{a}\,\frac{\sum_{n=1}^{\infty} A_n e^{-\lambda_n z^*} F_n'(1)}{\sum_{n=1}^{\infty} A_n e^{-\lambda_n z^*} \int_0^1 2(1-r^{*2})F_n(r^*)\,2r^*\,dr^*}$$

$$h\Big|_{z^*>>1} = \frac{k}{a}\,\frac{F_1'(1)}{\int_0^1 2(1-r^{*2})F_1(r^*)\,2r^*\,dr^*}$$

POD '12

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The Nusselt-Graetz Problem: Constant Temperature at the Wall

We solve the Sturm-Liouville problem for constant temperature at the wall for laminar flow through a circular tube.

The eigenvalue problem for the decaying solution is

((x*y')' + lambda*2*r*(1-r^2)*y = 0

y'(0) = 0; y(1) = 0

The asymptotic solution at large z is just Tinf = 1. Thus, the initial value for the Sturm-Liouville expansion is just -1.

Solving the problem we get:

```
p = @(x) x;
q = @(x) zeros(size(x));
w = @(x) 2*x.*(1-x.^2);
bc = [0,1,1,0];

n = 100; %The number of points we would like (the number of intervals)

[lambda, eigenvecs] = slsolve(p,q,w,bc,n);

Tinf = 1

% And that's it!
```

```
Tinf =

     1
```

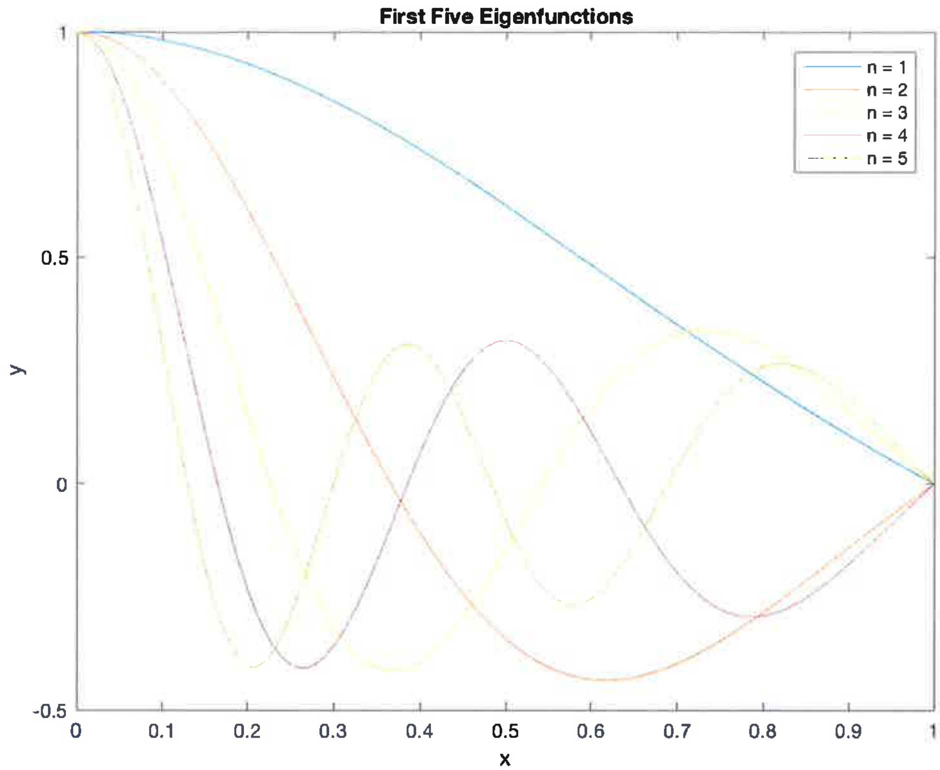
Eigenvalues, Eigenvectors, and Coefficients

We are interested in the lead eigenvalues, coefficients, and eigenvectors. We just look at the first five:

```
firsteigenvecs = lambda(1:5)

% And we calculate the coefficients using the Trapezoidal Rule:

r = [0:1/n:1]';
```

```
% The Trapezoidal Rule weights:
weights = ones(1,n+1);
weights(1) = 0.5;
weights(n+1) = 0.5;
weights=weights/n;

a = zeros(length(lambda),1);

for i = 1:length(lambda)
    numerator = -weights*(w(r).*Tinf.*eigenvecs(:,i));
    denominator = weights*(w(r).*eigenvecs(:,i).^2);
    a(i) = numerator/denominator;
end

firstcoefficients = a(1:5)

% And we plot the first five eigenfunctions:
figure(1)
plot(r,eigenvecs(:,1:5))
xlabel('x')
ylabel('y')
title('First Five Eigenfunctions')
legend('n = 1','n = 2','n = 3','n = 4','n = 5')
grid on
```

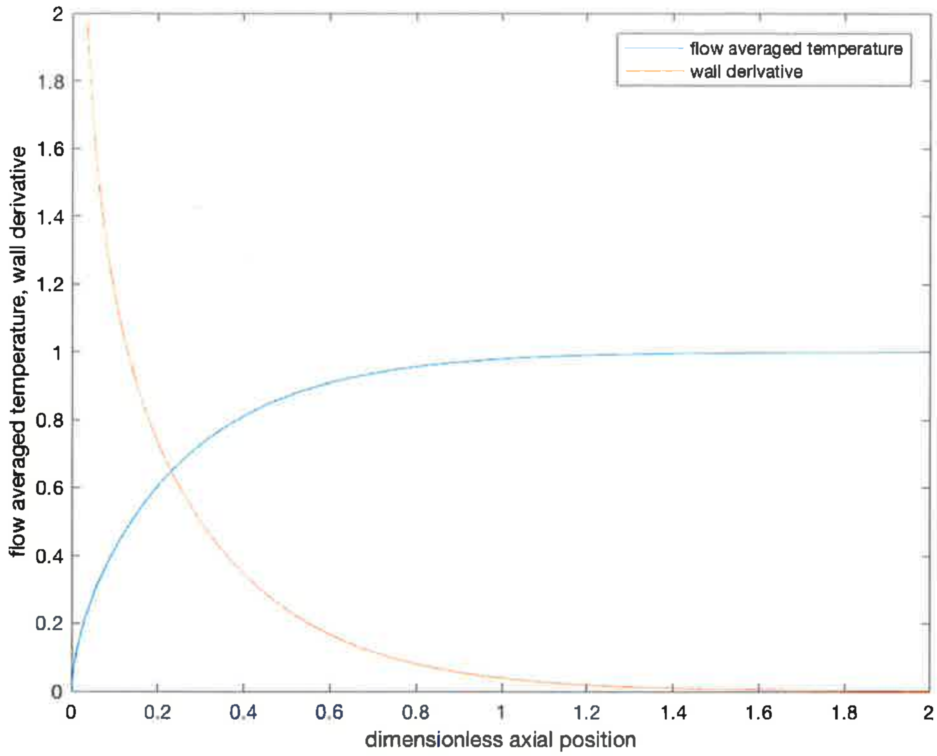
```
firsteigenvecs =

    3.6565
   22.2980
   56.9188
  107.4735
  173.9001


firstcoefficients =

   -1.4765
    0.8067
   -0.5885
    0.4768
   -0.4043
```



Flow Averaged Temperature and Heat Flux

To obtain the heat transfer coefficient we need to know the flow averaged temperature and the heat flux at the wall.

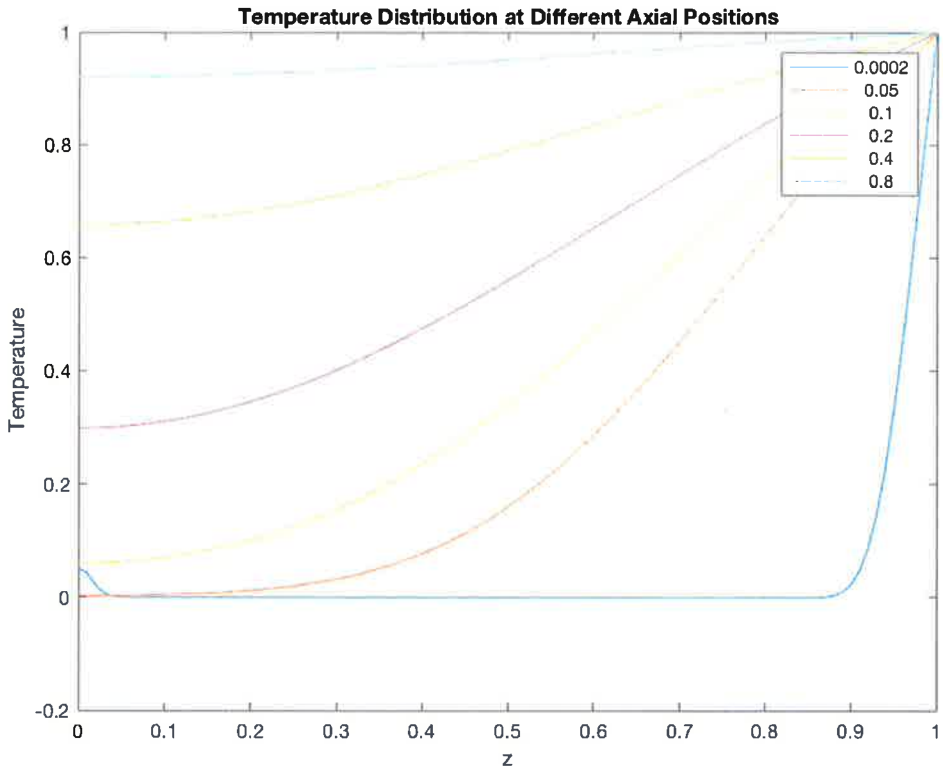
```
z = [0.0005:.001:2];
Tavg = zeros(size(z)); %We initialize the array
q = zeros(size(z));
T = zeros(size(r));
dr = 1/n;
for j = 1:length(z)
    for i=1:length(r)
        T(i) = Tinf + sum(a.*exp(-lambda*z(j)).*eigenvecs(i,:)');
    end
    Tavg(j) = weights*(2*(1-r.^2).*2.*r.*T);
    q(j) = (1.5*T(end)-2*T(end-1)+0.5*T(end-2))/dr;
end

figure(2)
plot(z,Tavg,z,q)
xlabel('dimensionless axial position')
ylabel('flow averaged temperature, wall derivative')
legend('flow averaged temperature','wall derivative')
axis([0 2 0 2])
grid on
```



Temperature Profile at Various Times

We can also plot up the temperature distribution for specific z locations. You will note the issue near the origin at very small z. This is known as the Gibbs ringing phenomenon and is well known in signal processing.

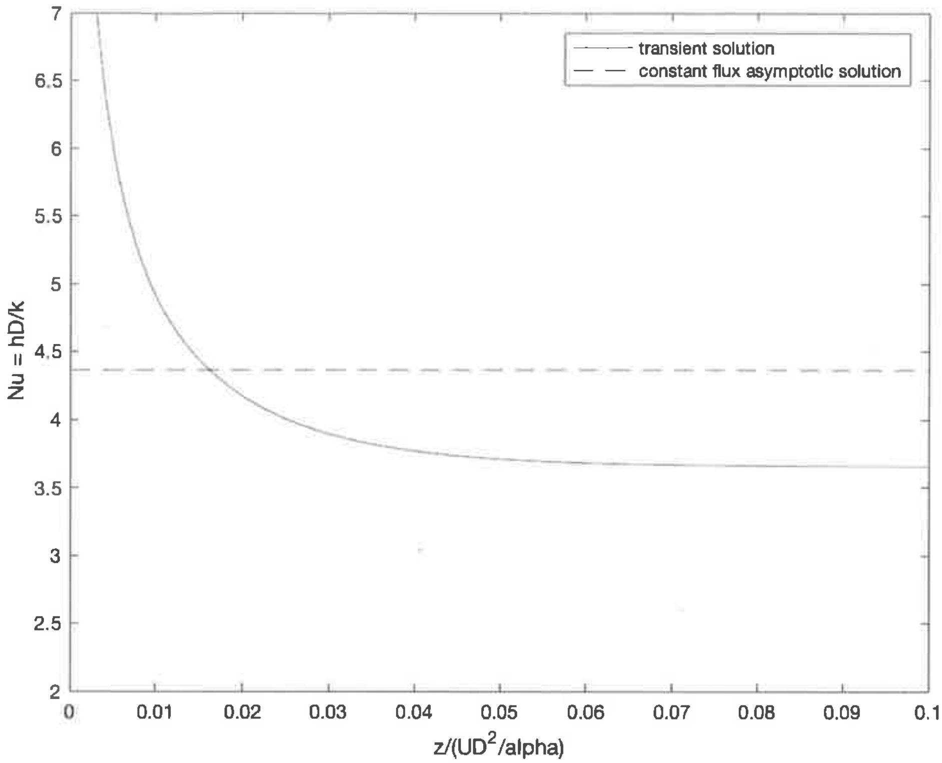
```
zplot = [0.0002,.05,.1,.2,.4,.8]';

tprofile = zeros(length(r),length(zplot));
for j = 1:length(zplot)
    for i=1:length(r)
        tprofile(i,j) = Tinf + sum(a.*exp(-lambda*zplot(j)).*eigenvecs(i,:)');
    end
end
figure(3)
plot(r,tprofile)
legend(num2str(zplot))
xlabel('z')
ylabel('Temperature')
title('Temperature Distribution at Different Axial Positions')
grid on
```



Heat Transfer Coefficient

Finally, we are interested in how the heat transfer coefficient varies with axial position. The dimensionless value is the ratio of the heat flux to the difference between the bulk and wall temperatures. So:

```
h = q./(1 - Tavg);
Nu = 2*h; %Convert to conventional definition hD/k

zstar = z/4; %Convert to conventional z/(UD^2/alpha)

figure(4)
plot(zstar,Nu,zstar,48/11*ones(size(zstar)),'--k')
axis([0,.1,2,7])
xlabel('z/(UD^2/alpha)')
ylabel('Nu = hD/k')
legend('transient solution','constant flux asymptotic solution')
grid on
```



Conclusion

The initial heat transfer coefficient is larger than the asymptotic value due to the small thickness of the developing boundary layer. After a short distance, however, the asymptotic result is reached. Both the heat transfer coefficient and the difference between wall and bulk temperature vanish, but the ratio is constant (and could be determined just from the lead eigenfunction). The heat transfer coefficient is slightly less than that obtained from the constant wall heat flux boundary condition. This is the usual value used in correlations for the Nusselt number.

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