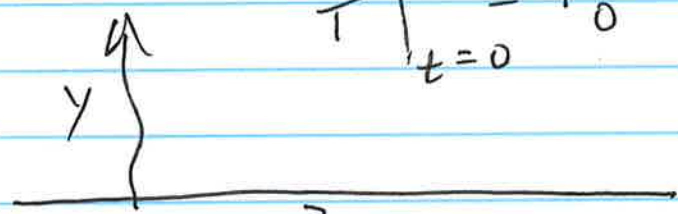
Problem of the Day 09

Semi-infinite domain w/ cst. heat flux

$$T\Big|_{t=0} = T_0$$



y

$$q_y\Big|_{\substack{t>0\\y=0}} = q_0 = -k\frac{\partial T}{\partial y}\Big|_{y=0}$$

So:

$$\rho \hat{C}_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial y^2}$$

$$T^* = \frac{T - T_0}{\Delta T_c} \qquad t^* = t/t_c \quad y^* = y/\delta$$

So:
$$\frac{\rho \hat{C}_p \Delta T_c}{t_c} \frac{\partial T^*}{\partial t^*} = \frac{k \Delta T_c}{\delta^2} \frac{\partial^2 T^*}{\partial y^{*2}}$$

Divide:

$$\frac{\delta^2 \rho \hat{C}_p}{t_c k} = \frac{\delta^2}{\alpha t_c}$$

$$\therefore \left[\frac{\delta^2}{\alpha t_c}\right] \frac{\partial T^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial y^{*2}}$$

$$\underset{1}{\shortparallel}$$

B.C.:

$$-\frac{k\Delta T_c}{\delta}\frac{\partial T^*}{\partial y^*}\bigg|_{y^*=0} = q_0$$

$$\text{or } \left[\frac{k\Delta T_c}{q_0\delta}\right]\frac{\partial T^*}{\partial y^*}\bigg|_{y^*=0} = -1$$

$$\underbrace{\left[\frac{k\Delta T_c}{q_0\delta}\right]}_{=1}$$

$$\text{So } \Delta T_c = \frac{q_0\delta}{k} \quad \left(= \frac{q_0(\alpha t_c)^{1/2}}{k}\right)$$

We've rendered eq'n dimensionless and never specified t_c! $\therefore$ will admit self-similar solution!

Check w/ stretching!

$$\frac{\partial T^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial y^{*2}}\,; \quad T^*\Big|_{y^*\to\infty} = 0, \quad T^*\Big|_{t^*=0} = 0$$

$$\frac{\partial T^*}{\partial y^*}\bigg|_{y^*=0} = -1$$

Let $T^* = A\bar{T}$, $t^* = B\bar{t}$, $y^* = C\bar{y}$

$$\frac{A}{B}\frac{\partial \bar{T}}{\partial \bar{t}} = \frac{A}{C^2}\frac{\partial^2 \bar{T}}{\partial \bar{y}^2}$$

Divide out!

$$\frac{\partial \bar{T}}{\partial \bar{t}} = \underbrace{\left[\frac{B}{C^2}\right]}_{=1}\frac{\partial^2 \bar{T}}{\partial \bar{y}^2}$$

Inhomogeneous BC: $\left.\frac{A}{C}\frac{\partial \bar{T}}{\partial \bar{y}}\right|_{\bar{y}=0} = -1$

$\therefore \frac{A}{C} = 1$

Canonical Form: complexity in t^* (B)

$\therefore \frac{C}{B^{1/2}} = 1 \quad \frac{A}{B^{1/2}} = 1$

$\therefore \frac{T^*}{t^{*1/2}} = f(\eta) \quad \eta = \frac{y^*}{t^{*1/2}}$

Now for D.E.:

$$T^* = t^{*1/2} f(\eta)$$

$$\therefore \frac{\partial T^*}{\partial y^*} = t^{*1/2} f' \frac{\partial \eta}{\partial y^*} = t^{*1/2} f' \frac{1}{t^{*1/2}} = f' !$$

B.C. : $f'(0) = -1$

$$\frac{\partial^2 T^*}{\partial y^{*2}} = \frac{\partial}{\partial y^*}(f') = f'' \frac{1}{t^{*1/2}}$$

Now for time deriv!

$$\frac{\partial T^*}{\partial t^*} = \frac{\partial}{\partial t^*}\left(t^{*1/2} f\right) = \frac{1}{2} t^{*-1/2} f + t^{*1/2} \frac{\partial f}{\partial t^*}$$

$$= \frac{1}{2} t^{*-1/2} (f - \eta f')$$

So the DE is:

$$f'' = \frac{1}{2}(f - \eta f')$$

$$f'(0) = -1, \quad f(\infty) = 0$$

and $$T = \frac{q_0 (\alpha t)^{1/2}}{k} f\left(\frac{y}{(\alpha t)^{1/2}}\right) + T_0$$

OK, how to get f? Easiest to do numerically, but we can actually get an analytic sol'n with a trick

Take derivative of ODE!

$$\frac{d}{d\zeta}\left\{ f'' = \frac{1}{2}(f - \zeta f') \right\}$$

$$\therefore f''' = \frac{1}{2}\left(\cancel{f'} - \cancel{f'} - \zeta f''\right)$$

$$\text{so } f''' = -\frac{1}{2}\zeta f''$$

Divide by f'':

$$\frac{f'''}{f''} \equiv \frac{d \ln f''}{d\zeta} = -\frac{1}{2}\zeta$$

$$\text{so } \ln f'' = -\frac{1}{4}\zeta^2 + cst$$

$$f'' = \frac{1}{2}f(0)\, e^{-\frac{1}{4}\zeta^2}$$

from original DE eval. at $\zeta = 0$

(we don't know $f(0)$)

We could integrate twice and get in terms of $\mathrm{ierfc}(x)$, the integral complementary error function...

Instead, just do it numerically!

$$\underset{\sim}{f} \equiv \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \equiv \begin{bmatrix} f \\ f' \end{bmatrix}$$

$$\frac{d\underset{\sim}{f}}{d\eta} = \begin{bmatrix} f_2 \\ +\frac{1}{2}(f_1 - \eta f_2) \end{bmatrix}$$

$$\underset{\sim}{f}(0) = \begin{bmatrix} x \\ -1 \end{bmatrix} \leftarrow \text{unknown}$$

Iterate until $f_1(\infty) = 0$!

~~Plot~~

After a second or two, we get $f_1(0) = \underline{1.1284}$ (a nice $O(1)$ number!)

We are actually interested in plotting lines of <u>constant</u> <u>temperature</u> ~~for different~~ as a function of time!

This will yield the melt zone!

Let T_m = melting temperature

$$\therefore T_m^* = \frac{T_m - T_0}{\Delta T_c}$$

We seek to determine y^* (or ζ) for which $T^* = T_m^*$ at a particular time t^*. This would be some ζ_m that is a function of t^*:

$$T_m^* = t^{*1/2} f(\zeta_m)$$

$$y^* = \zeta_m t^{*1/2}$$

Note that $f(\zeta)$ has a max of 1.128 so we <u>only</u> exceed T_m^* (and start to melt) if $t^* > \left(\frac{T_m^*}{f(0)}\right)^2$

or $t > \cancel{t_c} \left(\frac{T_m - T_0}{f(0)}\right)^2 \left(\frac{k}{q_0}\right)^2 \left(\frac{1}{\alpha \cancel{t_c}}\right)$

$$\text{or } t > \left(\frac{T_m - T_0}{f(0)}\right)^2 \frac{k^2}{q_0^2 \alpha}$$

After this time the melt zone would propagate through the slab!

Contents

Heated Semi infinite slab: Constant Wall Heat Flux

The miss2 program is:

function out = miss2(x) % This function takes in a guess for the derivative of the temperature at % y = 0 for an impulsively heated semi-infinite domain, but with heat flux % at the wall this time.

fdot = @(eta,f) [f(2); 0.5*(f(1)-eta*f(2))]; %The differential equation

f0 = [x,-1]; %The initial value

[etaout fout] = ode45(fdot,[0 10],f0);

out = fout(end,1);

```
x = 1; % Our initial guess for the temperature at the wall

x = fzero('miss2',x) % Our solution!
```

```
x =

    1.1284
```

Plotting things up

We just cut and paste from the miss.m routine to get the profile:

```
fdot = @(eta,f) [f(2); 0.5*(f(1)-eta*f(2))]; %The differential equation

f0 = [x,-1]; %The initial value

[etaout fout] = ode45(fdot,[0 10],f0);

out = fout(end,1);

figure(1)
plot(etaout,fout(:,1))
xlabel('eta')
ylabel('dimensionless temperature')
title('Temperature Distribution for Constant Heat Flux')
axis([0 5 0 1.5])
grid on
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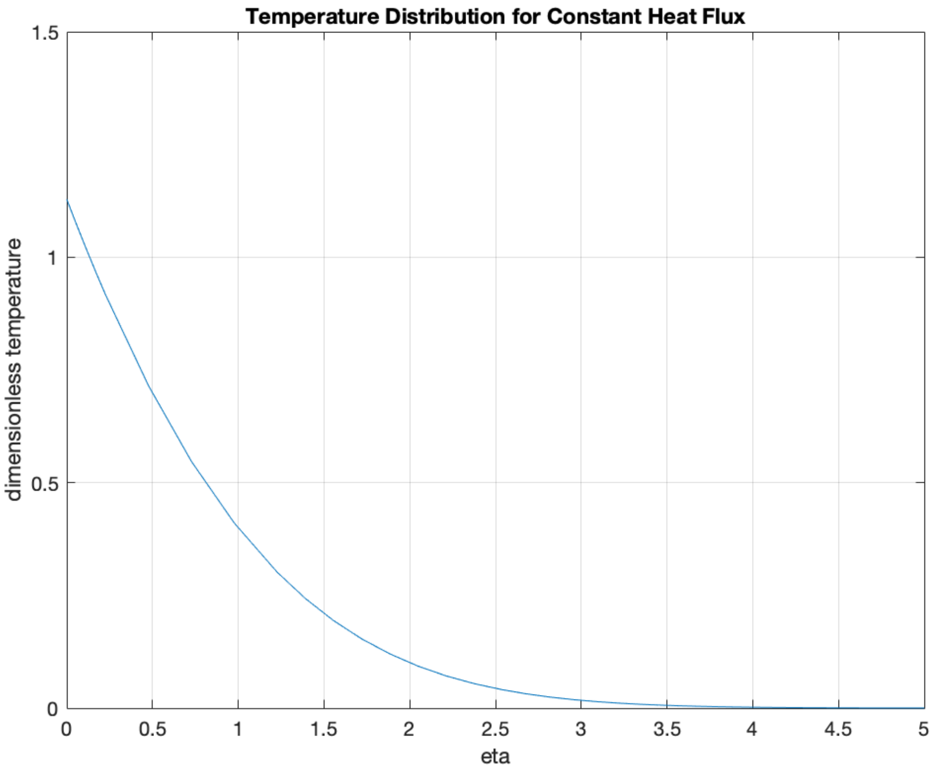

Published with MATLAB® R2017a