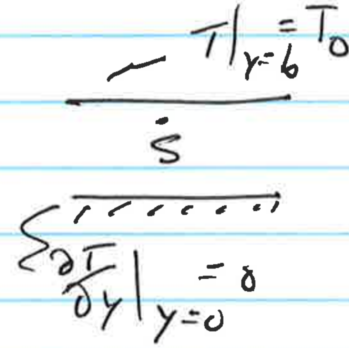
Problem of the Day Lecture 08

Demonstration of Neumann stability criterion

Slab w/ heat source, $T|_{y=b} = T_0$

Insulated at $y=0$



$$\therefore \quad \rho \hat{C}_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial y^2} + \dot{S}$$

$$y^* = y/b \qquad T^* = \frac{T - T_0}{\Delta T_c}$$

$$t^* = \frac{t}{t_c}$$

$$\frac{\rho \hat{C}_p \Delta T_c}{t_c} \frac{\partial T^*}{\partial t^*} = \frac{k \Delta T_c}{b^2} \frac{\partial^2 T^*}{\partial y^{*2}} + \dot{S}$$

Divide by cond. term:

$$\left(\frac{b^2}{\alpha t_c}\right) \frac{\partial T^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial y^{*2}} + \left[\frac{\dot{S} b^2}{k \Delta T_c}\right]$$

$$\left(\frac{b^2}{\alpha t_c}\right) = 1 \qquad \left[\frac{\dot{S} b^2}{k \Delta T_c}\right] = 1$$

$$\therefore \quad t_c = b^2/\alpha \qquad \Delta T_c = \frac{\dot{S} b^2}{k}$$

So:

$$\frac{\partial T^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial y^{*2}} + 1$$

$$T^*\Big|_{t^*=0} = 0 \quad T^*\Big|_{y^*=1} = 0 \quad \frac{\partial T^*}{\partial y^*}\Big|_{y^*=1} = 0$$

Let's get analytic sol'n to compare!

Need T_∞^*:

$$\cancel{\frac{\partial T_\infty^*}{\partial t^*}} = \frac{\partial^2 T_\infty^*}{\partial y^{*2}} + 1$$

0 at SS from BC at $y^*=1$ (sink)

$$\therefore T_\infty^* = -\frac{1}{2} y^{*2} + A y^* + B$$

$$\frac{\partial T_\infty^*}{\partial y^*}\Big|_{y^*=0} = 0 \quad \therefore \underline{A = 0}$$

$$T_\infty^*\Big|_{y^*=1} = 0 \quad \therefore \underline{B = \frac{1}{2}}$$

$$T_\infty^* = \frac{1}{2}\left(1 - y^{*2}\right)$$

$$T^* = T_\infty^* + T_d^*$$

$$\frac{\partial T_d^*}{\partial t^*} = \frac{\partial^2 T_d^*}{\partial y^{*2}} \qquad T_d^*\Big|_{y^*=1} = 0 \qquad \frac{\partial T_d^*}{\partial y^*}\Big|_{y^*=0} = 0$$

$$T_d^*\Big|_{t^*=0} = -T_\infty^*\Big|_{t^*=0} = -\frac{1}{2}\left(1 - y^{*2}\right)$$

$$T_d^* = G(t^*)F(y^*)$$

$$\therefore \quad \frac{G'}{G} = \frac{F''}{F} = -\sigma^2$$

$$G = e^{-\sigma^2 t^*} \qquad F'' + \sigma^2 F = 0$$

$$F(1) = 0 \quad F'(0) = 0$$

So $F = A\sin\sigma y^* + B\cos\sigma y^*$

$F'(0) = 0 \quad \therefore \underline{A = 0}$

$F(1) = 0 \quad \therefore \sigma = \left(n - \frac{1}{2}\right)\pi$

$$T_d^* = \sum_{n=1}^{\infty} B_n e^{-\sigma_n^2 t^*} \cos\sigma_n y^*$$

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$$B_n = \frac{\int_0^1 -\frac{1}{2}(1-y^{*2})\cos\sigma_n y^* dy^*}{\int_0^1 \cos^2\sigma_n y^* dy^*}$$

$$= \frac{2(-1)^n}{\pi^3 (n-\frac{1}{2})^3}$$

So:

$$T^*\Big|_{y^*=0} = T^*_\infty\Big|_{y^*=0} + T^*_d\Big|_{y^*=0}$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{\pi^3(n-\frac{1}{2})^3} e^{-((n-\frac{1}{2})\pi)^2 t^*}$$

OK, now for numerics!

$$\frac{\partial^2 T_i^*}{\partial y^{*2}} \approx \frac{T_{i-1}^* - 2T_i^* + T_{i+1}^*}{\Delta y^{*2}}$$

$$\therefore T_i^{k+1} \approx T_i^k + \Delta t \left[\frac{T_{i-1}^k - 2T_i^k + T_{i+1}^k}{\Delta y^2} + 1 \right]$$

$T_n^{K+1} = 0$ (upper BC)

Lower BC: $\left.\frac{\partial T}{\partial y^*}\right|_{y=0}^{*} \approx \frac{T_1^* - T_0^*}{\Delta y^*} - \frac{1}{2}\Delta y^* \frac{T_2^* - 2T_1^* + T_0^*}{\Delta y^2}$

$$= \frac{1}{\Delta y^*}\left[2T_1^* - \frac{3}{2}T_0^* - \frac{1}{2}T_2^*\right] = 0$$

$$\therefore \frac{3}{2}T_0^* = 2T_1^* - \frac{1}{2}T_2^*$$

$$T_0^* = \frac{4}{3}T_1^* - \frac{1}{3}T_2^*$$

So $T_0^{K+1} = \frac{4}{3}T_1^{K+1} - \frac{1}{3}T_2^{K+1}$

which is imposed after updating interior nodes!

Neumann condition: $\Delta t^* < \frac{1}{2}\Delta y^{*2}$

Problem of the Day 08: Neumann Stability - Slab with heat generation

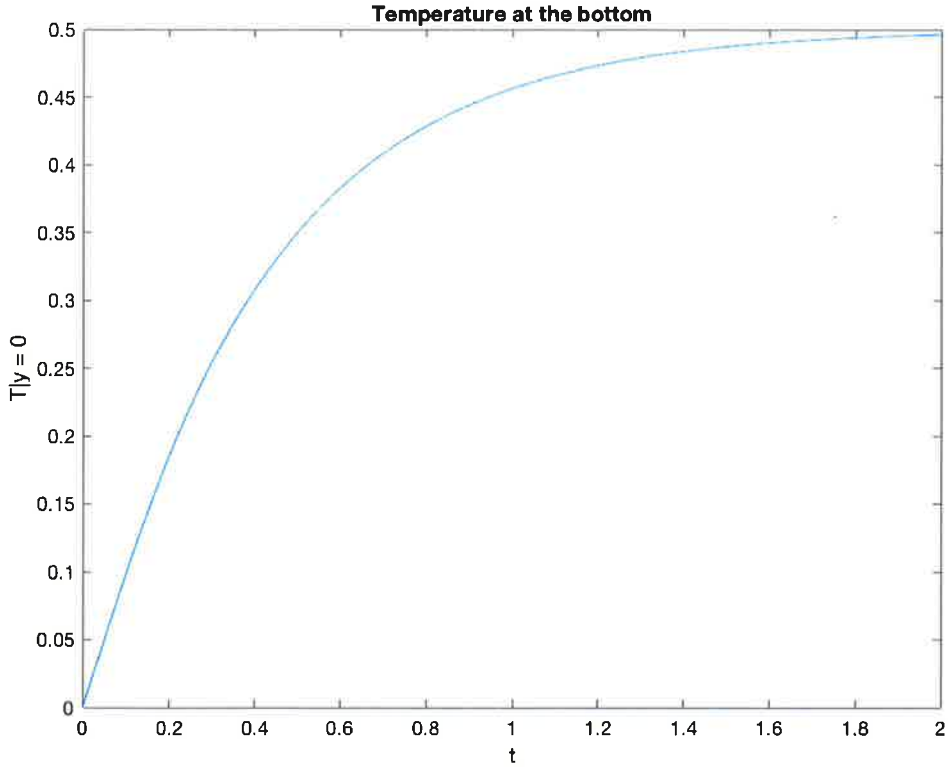
In this problem we compare a finite difference marching solution to the exact result for a slab with uniform heat generation. We are interested in the temperature at the bottom (insulated wall) as a function of time. The problem admits a nice closed form SL solution, so this gives us a point of comparison for the numerical result. We look at time spacings just above and below the Neumann stability criterion.

Contents

The exact solution:

We have the Sturm Liouville solution:

```
tsl = [0:.001:2];
n = [1:100]'; % we use 100 eigenvalues (overkill)
sigma = (n-.5)*pi; % the eigenvalues

Tbotsl = 1/2 + sum(2*(-1).^n./sigma.^3.*exp(-sigma.^2*tsl));

figure(1)
plot(tsl,Tbotsl)
xlabel('t')
ylabel('T|y = 0')
title('Temperature at the bottom')
grid on
```

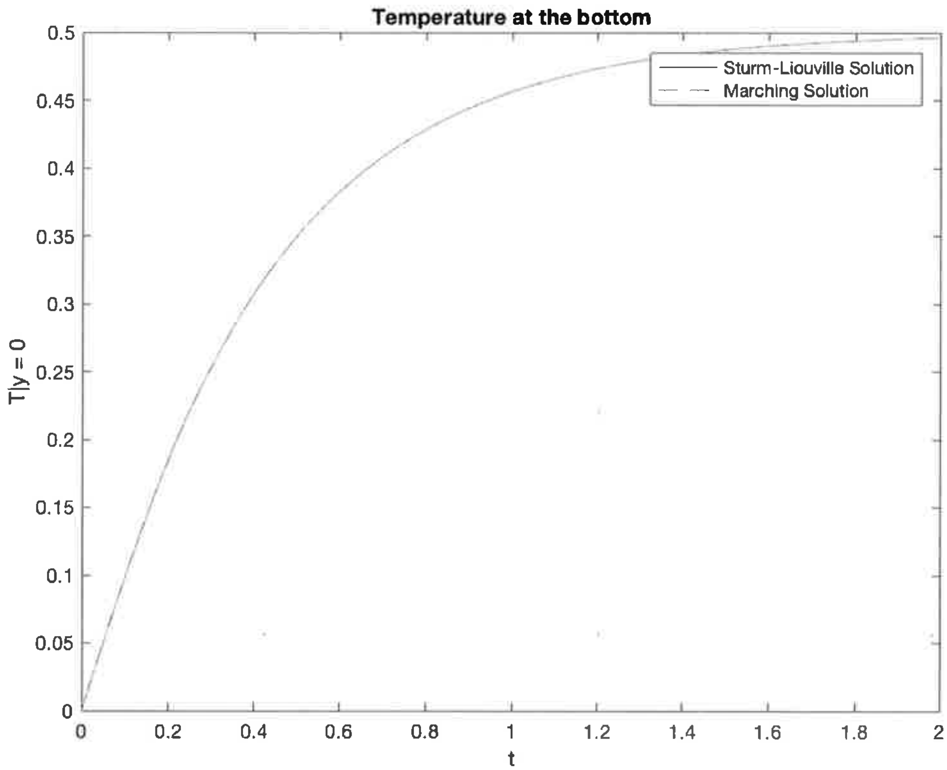


The marching solution below the Neumann condition

We use the center difference Euler method marching solution. We choose a discretization of n in the spatial domain and thus have a time discretization of less than 0.5/n^2 for stability:

```
n = 20
dy = 1/n;

a = (diag(ones(n,1),-1)+diag(ones(n,1),1)-2*diag(ones(n+1,1)))/dy^2;

format short e
dt = 0.50*dy^2 %The maximum dt for stability

tkeep = [0:dt:3]; %The times we keep
Tbot = zeros(size(tkeep)); %We initialize the bottom temperature array

Tbot(1) = 0;
T = zeros(n+1,1); %our initial temperature distribution

for i = 2:length(tkeep)
    T = T + dt * (a * T + 1); %We add in the source
    T(n+1) = 0; %The upper BC
    T(1) = 4/3*T(2) - 1/3*T(3); %The lower BC
    Tbot(i) = T(1); %We keep the bottom temperature
end

figure(1)
hold on
plot(tkeep,Tbot,'--')
hold off
legend('Sturm-Liouville Solution','Marching Solution')
axis([0 2 0 .5])
```

```
dt =

   1.2500e-03
```

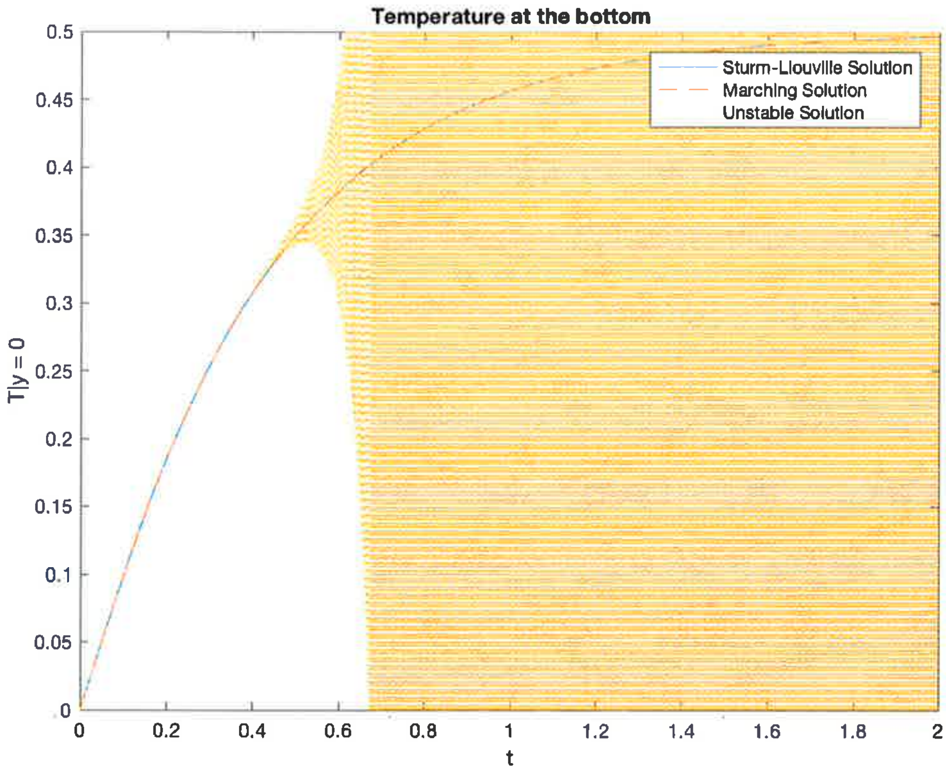


Now we increase dt by just a bit...

```
dt = 0.51*dy^2 %just above stability!
format short

Tbotnew = zeros(size(tkeep)); %We initialize the bottom temperature array

Tbotnew(1) = 0;
T = zeros(n+1,1); %our initial temperature distribution

for i = 2:length(tkeep)
    T = T + dt * (a * T + 1); %We add in the source
    T(n+1) = 0; %The upper BC
    T(1) = 4/3*T(2) - 1/3*T(3); %The lower BC
    Tbotnew(i) = T(1); %We keep the bottom temperature
end

figure(1)
hold on
plot(tkeep,Tbotnew,':')
hold off
legend('Sturm-Liouville Solution','Marching Solution','Unstable Solution')
axis([0 2 0 .5])
```

```
dt =
```

```
1.2750e-03
```



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