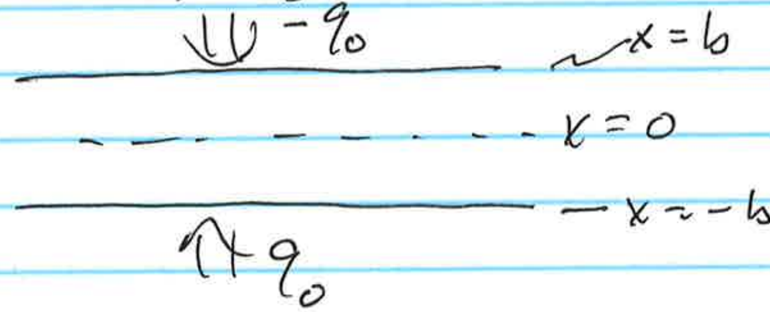
Lecture 06 P.O.D: ①

time-dependent asymptotic sol'n

Usually asymptotic solution is <u>steady</u>, but not always! You can have a time dep. BC at wall, or a heat flux cond. w/ no way for energy to get away! Thus, asymptotic sol'n will <u>still</u> grow in time!



Heat slab from <u>both</u> sides!

So: $$\rho \hat{C}_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \quad \text{(symmetry)}$$

$$-k \left.\frac{\partial T}{\partial x}\right|_{x=b} = -q_0 \qquad T\Big|_{t=0} = T_0$$

Let's render dimensionless!

$$T^* = \frac{T - T_0}{\Delta T_c} \qquad x^* = \frac{x}{b} \qquad t^* = t/t_c$$

$$\therefore \quad \frac{\rho \hat{C}_p \cancel{\Delta T_c}}{t_c} \frac{\partial T^*}{\partial t^*} = k \frac{\cancel{\Delta T_c}}{b^2} \frac{\partial^2 T^*}{\partial x^{*2}}$$

so $t_c = b^2/\left(\frac{k}{\rho \hat{C}_p}\right) \equiv b^2/\alpha$ as usual!

$$\frac{\Delta T_c k}{b} \left.\frac{\partial T^*}{\partial x^*}\right|_{x^*=1} = q_0 \qquad \therefore \Delta T_c = \frac{q_0 b}{k}$$

So: $$\frac{\partial T^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial x^{*2}}$$

$$\left.\frac{\partial T^*}{\partial x^*}\right|_{x^*=0} = 0 \qquad \left.\frac{\partial T^*}{\partial x^*}\right|_{x^*=1} = 1 \qquad T^*\Big|_{t^*=0} = 0$$

We anticipate that T_∞^* will grow linearly in time! To satisfy DE we also have to have a time indep. part!

$\therefore$ Let $T_\infty^* = t^* f_1(x^*) + f_2(x^*)$

Plug in:

$$\frac{\partial T_\infty^*}{\partial t^*} = \frac{\partial}{\partial t^*}\left(t^* f_1 + f_2\right) = f_1$$

and

$$\frac{\partial^2 T_\infty^*}{\partial x^{*2}} = t^* \frac{\partial^2 f_1}{\partial x^{*2}} + \frac{\partial^2 f_2}{\partial x^{*2}}$$

So:

$$f_1 = t^* f_1'' + f_2''$$

We need this to work at all t^*

$\therefore$ we get two problems:

$$f_1'' = 0 \qquad f_1'(0) = 0 \quad f_1'(1) = 0$$

and $\quad f_2'' = f_1 \qquad f_2'(0) = 0 \quad f_2'(1) = 1$

BC

The solution to f_1 is a constant!

$f_1 = c$

$\therefore f_2'' = c \qquad f_2' = c\,x^* + A$

$f_2'(0) = 0 \quad \therefore A = 0$

$f_2'(1) = 1 \quad \therefore c = 1$!

finally, $f_2 = \frac{1}{2} x^{*2} + B$

But there are no more BC's??

we deal with this by an integral energy balance

At $t^* = 0$ we know that the average temp. is zero!

$$T_\infty^* = t^* + \frac{1}{2} x^{*2} + B$$

Let's take this T_∞^* to satisfy avg. at $t^* = 0$!

$$\int_0^1 T_\infty^*\Big|_{t^*=0} dx^* = 0 = \int_0^1 \left(\frac{1}{2}x^{*2} + B\right) dx^*$$

$$= \frac{1}{6} + B \qquad \therefore B = -\frac{1}{6}$$

and $T_\infty^* = t^* + \frac{1}{2}x^{*2} - \frac{1}{6}$

Note that at all times the wall is $\Delta T^* = \left(\frac{1}{2} - \frac{1}{6}\right) - \left(-\frac{1}{6}\right) = \frac{1}{6}$ hotter than the centerline!

OK, now for the decaying sol'n:

$$T^* = T_\infty^* + T_d^*$$

$$\frac{\partial T_d^*}{\partial t^*} = \frac{\partial^2 T_d^*}{\partial x^{*2}} \qquad \frac{\partial T_d^*}{\partial x^*}\Big|_{x^*=0,1} = 0$$

$$T_d^*\Big|_{t^*=0} = -T_\infty^*\Big|_{t^*=0} = -f_2(x^*)$$

Use separation of variables:

$$T_Q^* = G(t^*)F(x^*)$$

$$\therefore \frac{G'}{G} = \frac{F''}{F} = -\sigma^2$$

$$G = e^{-\sigma^2 t^*}$$

$$F'' + \sigma^2 F = 0 \ ; \ F'(0) = F'(1) = 0$$

$$F = A\sin\sigma x^* + B\cos\sigma x^*$$

$$F'(0) = 0 \ \therefore A = 0$$

$$F'(1) = 0 \ \therefore B\sigma\sin\sigma = 0$$

so $\sigma = n\pi$

Note that there is a zero eigenvalue

This wouldn't decay!

Because we've correctly constructed T_∞^*

this has to vanish!

$$T_d^* = \sum_{n=1}^{\infty} B_n e^{-n^2\pi^2 t^*} \cos n\pi x^*$$

Get B_n from orthogonality:

$$B_n = \frac{\int_0^1 -f_2(x^*) \cos n\pi x^* \, dx^*}{\int_0^1 \cos^2 n\pi x^* \, dx^*}$$

$\frac{1}{2} =$

and $\int_0^1 \left(\frac{1}{6} - \frac{1}{2} x^{*2}\right) \cos n\pi x^* \, dx^* = \frac{-(-1)^n}{(n\pi)^2}$

so $T^* = t^* + \frac{1}{2} x^{*2} - \frac{1}{6} + \sum_{n=1}^{\infty} 2 \frac{-(-1)^n}{(n\pi)^2} e^{-n^2\pi^2 t^*} \cos n\pi x^*$

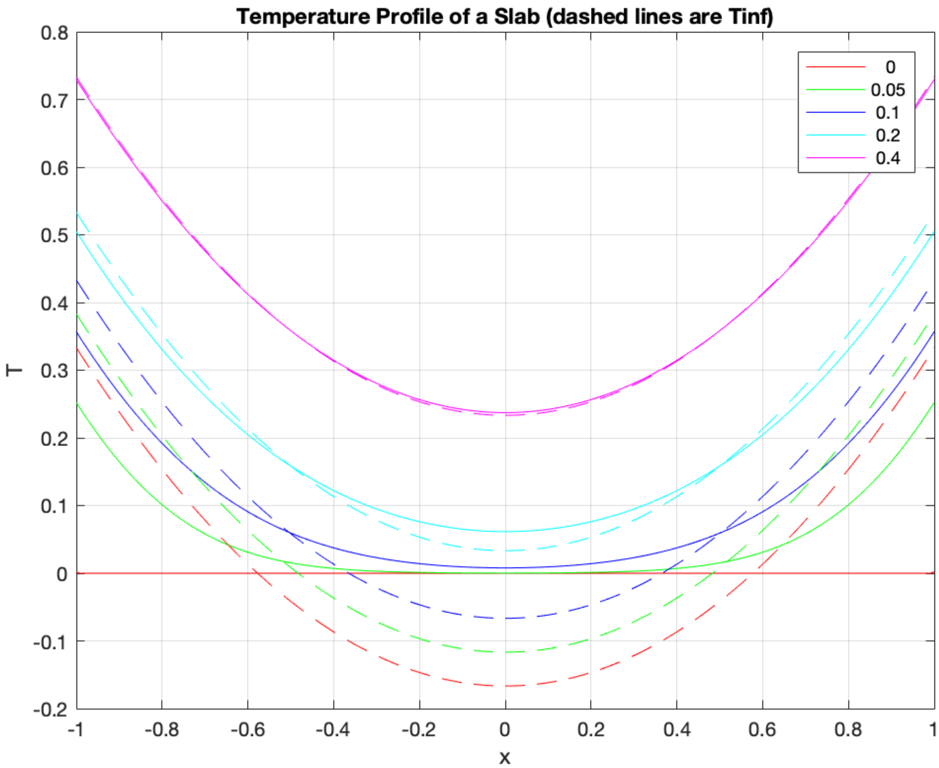
Because the lead eigenvalue is π, the decaying solution vanishes as $e^{-\pi^2 t^*}$, so by $t^* \gtrsim \frac{1}{4}$ or so $T^* \simeq T_\infty^*$. That's why you need to know σ_1!

Problem of the Day 06: Heating of a Slab from Both Sides

We plot up the dimensionless profile of the temperature distribution for a slab with heating at both sides. We take t to be a scalar and x to be a row vector. That way the product of a column vector n and row vector x produces a matrix which can be summed down the columns (the default for matlab) to sum the series without using a loop. Because the lead eigenvalue is pi, the decaying solution vanishes as exp(-pi^2*t), so it is all over by a dimensionless time of around 0.25 or so.

```
Tinf = @(t,x) t + 0.5*x.^2 - 1/6

n = [1:100]'; %We keep 100 eigenvalues (overkill)
Td = @(t,x) sum(2*(-(-1).^n./(n*pi).^2.*exp(-(n*pi).^2*t))*ones(size(x)).*cos(n*pi*x))

x = [-1:.01:1];

figure(1)
tall = [0,.05,.1,.2,.4]';
colors = 'rgbcmyk';

for i = 1:length(tall)
    plot(x,Tinf(tall(i),x)+Td(tall(i),x),colors(i))
    hold on
end
for i = 1:length(tall)
    plot(x,Tinf(tall(i),x),[colors(i),'--'])
    hold on
end

hold off
xlabel('x')
ylabel('T')
legend(num2str(tall))
grid on
title('Temperature Profile of a Slab (dashed lines are Tinf)')
```

```
Tinf =

  function_handle with value:

    @(t,x)t+0.5*x.^2-1/6


Td =

  function_handle with value:

    @(t,x)sum(2*(-(-1).^n./(n*pi).^2.*exp(-(n*pi).^2*t))*ones(size(x)).*cos(n*pi*x))
```



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