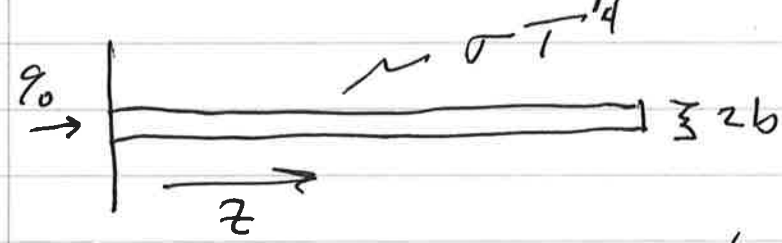
Lecture 04 Problem of the Day

Radiation fin in space!



- What is the temperature at the base of the fin?

- How long should it be?

$$0 = k \frac{\partial^2 T}{\partial z^2} - \frac{\sigma T^4}{b}$$ (no back radiation)

↳ assume avg temp is surface temp

so this is $\frac{\text{loss}}{\text{volume}}$!

$$q_0 = -k \frac{\partial T}{\partial z}\Big|_{z=0}$$

Let's scale this

$$T^* = \frac{T}{\Delta T_c} \qquad z^* = \frac{z}{z_c}$$

we want to know ΔT_c & z_c!

(2)

Plug in:

$$0 = \frac{k \Delta T_c}{z_c^2} \frac{\partial^2 T^*}{\partial z^{*2}} - \frac{\sigma \Delta T_c^4}{b} T^{*4}$$

Divide: $$0 = \frac{\partial^2 T^*}{\partial z^{*2}} - \left[\frac{\sigma \Delta T_c^3 z_c^2}{b k}\right] T^{*4}$$

$$\therefore \frac{\sigma \Delta T_c^3 z_c^2}{b k} = 1$$

$$-k \frac{\Delta T_c}{z_c} \frac{\partial T^*}{\partial z^*}\bigg|_{z^*=0} = q_0$$

$$\therefore \frac{\partial T^*}{\partial z^*}\bigg|_{z^*=0} = -\underbrace{\left[\frac{q_0 z_c}{k \Delta T_c}\right]}_{=1}$$

So we have 2 eq'ns for ΔT_c & z_c!

$$\Delta T_c = \frac{q_0 z_c}{k}$$

$$\frac{\sigma q_0^3 z_c^5}{b k^4} = 1$$

$$z_c = \left(\frac{k^4 b}{\sigma q_0^3}\right)^{1/5} \qquad \Delta T_c = \frac{q_0}{k}\left(\frac{b k^4}{\sigma q_0^3}\right)^{1/5}$$

$$= \left(\frac{q_0^2 b}{\sigma k}\right)^{1/5}$$

Let's check units:

$$[q_0] = \frac{W}{m^2} \qquad [k] = \frac{W}{m\,{}^\circ K} \qquad [b] = m$$

$$[\sigma] = \frac{W}{m^2\,{}^\circ K^4}$$

$$\therefore \left[\frac{q_0^2\, b}{\sigma k}\right] = \frac{\cancel{W^2}/\cancel{m^4}\;\cancel{m}}{\dfrac{\cancel{W}}{\cancel{m^2}\,{}^\circ K^4}\;\dfrac{\cancel{W}}{\cancel{m}\,{}^\circ K}} = {}^\circ K^5$$

which checks!

OK, now for the dimensionless eq'n:

$$\frac{\partial^2 T^*}{\partial z^{*2}} = +T^{*4}$$

$$\left.\frac{\partial T^*}{\partial z^*}\right|_{z^*=0} = -1 \qquad \left.\frac{\partial T^*}{\partial z^*}\right|_{z^*\to\infty} = 0$$

This is a <u>non-linear</u> problem and must be solved <u>numerically</u>!

Let $T^* \equiv f_1$ $\quad\therefore\quad$ $f_1' = f_2$

$\frac{\partial T^*}{\partial z^*} \equiv f_2$ $\qquad$ $f_2' = f_1^4$

$f_2(0) = -1$ $\qquad$ $f_1(0) = x$ (unknown)

$f_2(\infty) = 0$

We want to find x! $\left(T^*\big|_{z^*=0}\right)$

From numerical solution:

$$T^*\Big|_{z^*=0} = 1.201 \text{ (nicely } O(1)!)$$

$(T^* \approx 0.5$ at $z^* = 2.2$ as well$)$

For SNAP-27 what is ΔT_c?

Assume $\frac{1}{16}$" steel, so:

$k = 45 \frac{W}{m\,°K}$ $\qquad$ $2b = 0.159\text{ cm}$

$$q_0 = \frac{1417\text{ W}}{(0.42)(0.00159)\,8} = 2.65 \times 10^5 \frac{W}{m^2}$$

\# of fins

$$\sigma = 5.67 \times 10^{-8} \frac{W}{m^2\,°K^4}$$

(5)

$$\therefore \Delta T_c = \left(\frac{(2.65\times10^5)^2 \left(\frac{0.00159}{2}\right)}{5.67\times10^{-8} \quad 45} \right)^{1/5}$$

$$= 466\ ^\circ K$$

and w/ $T^*\Big|_{z^*=0} = 1.2$

$$T\Big|_{z=0} = 1.2\,\Delta T_c = 559\ ^\circ K$$

This is close to the reported value of $547^\circ K$...

What is z_c?

$$z_c = \left(\frac{k^4 b}{\sigma\, q_o^3} \right)^{1/5} = \left(\frac{(45)^4 \left(\frac{0.00159}{2}\right)}{5.67\times10^{-8} (2.65\times10^5)^3} \right)^{1/5}$$

$$= 0.079\ m \cong 8\ cm$$

and the fins are a couple of times this length...

A more sophisticated model includes back radiation and emissivity!

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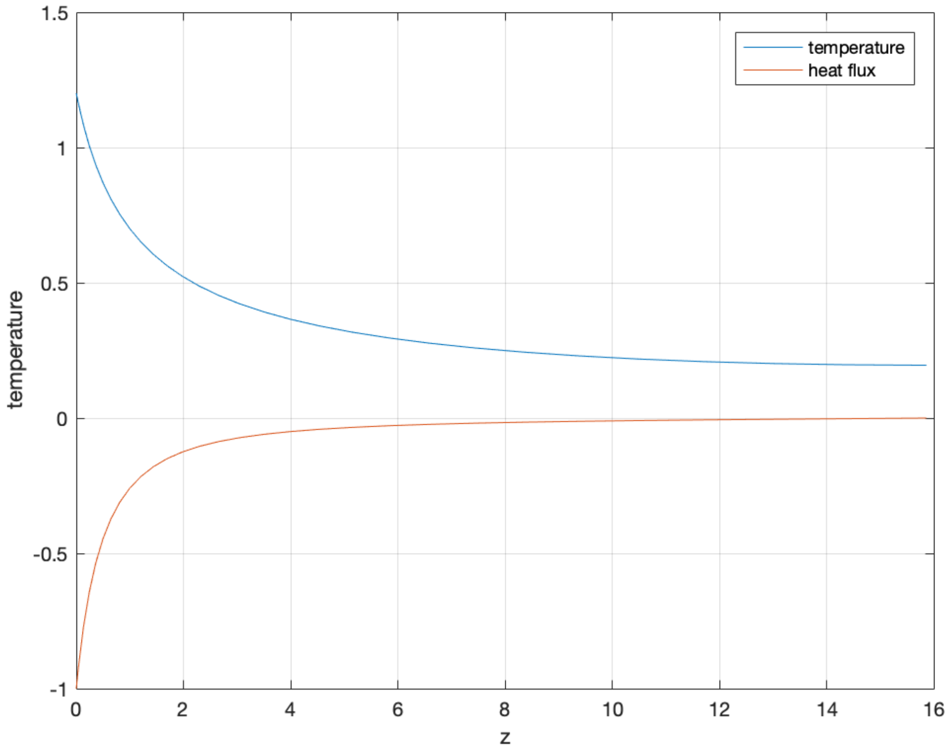
Solution to Fin Radiation Problem

In this script we solve the temperature distribution in a fin where the heat transfer from the fin is in the form of radiation. We assume that there is a specified heat flux at the base of the fin. Because of the strong dependence of the radiative heat flux on temperature the solution is very dependent on the initial condition - the temperature at z = 0. We thus use an iterative solution, progressing to larger values of z to achieve convergence.

```
global zlimpass

zlimpass = 1;
x = 1;
for i = 1:30
    x = fzero('miss',x);
    zlimpass = zlimpass*1.1;
end
initialtemp = x
```

```
initialtemp =

    1.2011
```



Conclusion

The final expression converges to the initial condition (temperature) of 1.2011, and the temperature of the fin drops to 0.5 at a distance of 2.2, both nicely O(1) quantities. The decrease in temperature slows drastically at larger z because of the T^4 dependence. At this position the dimensionless heat flux is only -0.11, which means that 89% of the total heat loss occurs for z less than 2.2. Thus, the rest of the fin is of little use: the optimal fin length should be about 1 - comprising 74% of the total heat loss. If we include back radiation (which becomes increasingly significant as the fin temperature drops) this further shortens the optimal length of the fin.

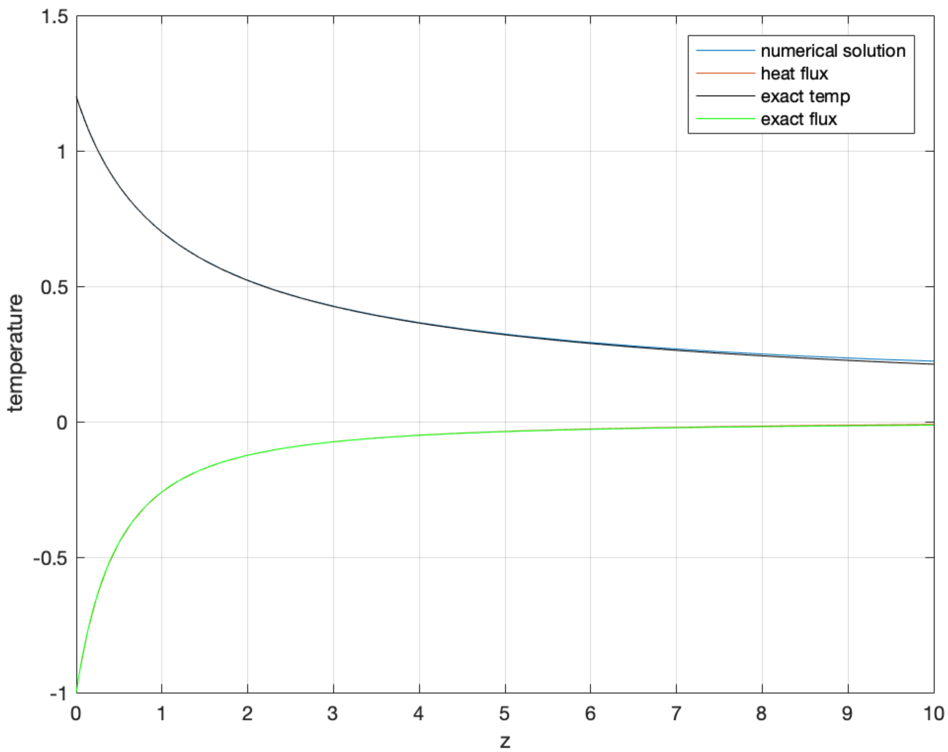
Comparison to Exact Solution

It turns out that you -can- solve this problem analytically: multiplying both sides by dT/dz you can get a perfect differential on both sides of the equation. Integrating and applying the boundary conditions you get a simple analytic solution. The deviation from the numerical solution is very small for small z, with the temperature deviating slightly at large z. We can add this to our figure:

```
T = @(z) (3*z/10^.5 + (2/5)^(3/10)).^(-2/3)

dTdz = @(z) -(2/5)^.5*T(z).^2.5

z = [0:.01:zlimpass];

figure(1)
hold on
plot(z,T(z),'k',z,dTdz(z),'g')
hold off
axis([0 10 -1 1.5])
legend('numerical solution','heat flux','exact temp','exact flux')
```

```
T =

  function_handle with value:
```

```
    @(z)(3*z/10^.5+(2/5)^(3/10)).^(-2/3)


dTdz =

  function_handle with value:

    @(z)-(2/5)^.5*T(z).^2.5
```



Miss.m function

The function called by the program is given below. Uncomment it and save it in a file named miss.m

```
% function out = miss(x)
% %This function takes in the initial value of the temperature at z = 0,
% %performs the integration to zlimpass (a global variable) and returns the
% %value of the derivative at zlimpass.  We then use this with a rootfinder
% %to get the correct value for x.
%
% global zlimpass
%
% fdot = @(z,f) [sign(f(1))*f(2);f(1)^4]; %The derivatives
% % Note that we have added a little "fix" to fdot, as things go haywire if
% % the temperature becomes negative as can happen if you have the wrong
% % initial condition.  Flipping the sign of the derivative forces stability
% % of the differential equation for these conditions: it is just a numerical
% % fix dealing with errors caused by the incorrect IC guess.
%
% [zout,fout] = ode23(fdot,[0 zlimpass],[x,-1]);
% out = fout(end,2); % we require the temperature derivative to be zero at zlimpass.
```

```
%
% % We add in a little graphics to see how we are doing.
% figure(1)
% plot(zout,fout)
% legend('temperature','heat flux')
% xlabel('z')
% ylabel('temperature')
% grid on
% zoom on
% drawnow
```

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