CBE 30355 Transport I Midterm Exam

October 16, 2025

1 Solution: Momentum balance in the wake-traverse experiment

Assumptions and conventions

- Steady, incompressible flow of density ρ.
- Flow is two-dimensional in the x-y plane; quantities given are *per unit length* in the z-direction (into the page). We will therefore quote forces as F/L.
- Inlet (left) velocity is uniform U across the channel height $2h$. Outlet (right) velocity is $u(y)$ (non-uniform) for $-h \le y \le h$.
- Inlet pressure is p_0 (constant on the inlet face); outlet pressure is p_e (constant on the outlet face).
- Side (top/bottom) friction and shear on the dashed control-surface walls are neglected. Body forces (gravity) are neglected in the x-momentum balance.
- Define F_{cyl} as the force exerted *by the cylinder on the fluid* in the positive x-direction. By Newton's third law the force exerted by the fluid on the cylinder (drag on the cylinder) is $F = -F_{cyl}$. We will present the final expression for F/L.

a. Mass (volume) conservation.

Steady incompressible mass conservation (per unit depth) gives

$$U(2h) = \int_{-h}^{h} u(y)\, dy.$$

This expresses that the volumetric flow rate per unit depth at the inlet equals that at the outlet.

b. Integral momentum balance (in words).

For the chosen control volume (dashed box surrounding the cylinder): "The net rate of x-momentum flux leaving the control volume minus the rate entering equals the sum of the external forces acting on the fluid inside the control volume in the x-direction." The external forces here are pressure forces on the inlet and outlet faces and the mechanical force the cylinder exerts on the fluid (we neglect wall shear on top/bottom and any body forces in x). In steady flow the momentum accumulation term is zero.

c. Integral expression for the drag per unit length F/L.

Write the integral x-momentum balance for steady flow. Sum of forces on the control volume in the x-direction equals the net outflow of x-momentum:

$$\sum F_x = \rho \left(\int_{A_{out}} u^2 \, dA - \int_{A_{in}} U^2 \, dA \right).$$

Evaluate each term (unit depth into the page so $dA = dy$):

- Pressure force on the inlet face (left): the external pressure p_0 pushes the fluid to the right, giving a positive contribution

$$F_{p,in} = p_0(2h).$$

- Pressure force on the outlet face (right): the external pressure p_e acts on the right face pushing left, so its contribution to the sum-of-forces (positive to the right) is

$$F_{p,out} = -p_e(2h).$$

- Force of the cylinder on the fluid (unknown): F_{cyl} (positive to the right by sign convention).
- No other x-directed forces (top/bottom wall shear neglected).
- Momentum flux terms:

$$\int_{A_{out}} u^2 \, dA = \int_{-h}^{h} u(y)^2 \, dy, \qquad \int_{A_{in}} U^2 \, dA = U^2(2h).$$

Thus the momentum balance becomes

$$p_0(2h) - p_e(2h) + F_{cyl} = \rho \left(\int_{-h}^{h} u(y)^2 \, dy - 2h \, U^2 \right).$$

Solve for F_{cyl}:

$$F_{cyl} = \rho \left(\int_{-h}^{h} u(y)^2 \, dy - 2h \, U^2 \right) - 2h \, (p_0 - p_e).$$

Recall the drag on the cylinder (the force exerted *by the fluid on the cylinder*) is $F = -F_{cyl}$. Therefore the drag per unit length is

$$\boxed{\frac{F}{L} = 2h \, (p_0 - p_e) + \rho \Big(2h \, U^2 - \int_{-h}^{h} u(y)^2 \, dy \Big)}.$$

Remarks

- The first term, $2h(p_0 - p_e)$, is the net pressure contribution (inlet minus outlet) acting on the control-volume faces. The second term is the difference in streamwise momentum flux (inlet minus outlet).

- Using mass conservation $2h\,U = \int_{-h}^{h} u(y)\,dy$ one can manipulate the momentum-flux term if desired, but in general the nonuniform $u(y)$ is left as the integral shown above.
- Sign interpretation: with the boxed convention F/L is the force of the fluid on the cylinder in the positive x-direction. If $F/L > 0$ the fluid pushes the cylinder downstream; typically for a fixed cylinder in a rightward flow the drag resisting the flow corresponds to a negative x-force on the fluid so the fluid-on-cylinder drag (as commonly reported) will have the opposite sign depending on the chosen sign convention.

2 Flow Resistance in Membranes: Unidirectional Flows

Assumptions. Steady, incompressible, axisymmetric, fully-developed laminar flow in straight cylindrical pores of length (membrane thickness) h. Entrance/exit and pore–pore interaction effects are neglected. Fluid viscosity is μ (subscripted where needed). The membrane porosity (open area fraction) is ϵ.

a. Flow through a single cylindrical pore (from Navier–Stokes).

Start from the z-momentum equation in cylindrical coordinates. Under the stated assumptions (steady, axisymmetric, no swirl, fully developed $\partial/\partial z = 0$ for u_z except via pressure gradient, and negligible body forces)

$$0 = -\frac{dp}{dz} + \mu\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{du_z}{dr}\right)\right],$$

where $u_z(r)$ is the axial velocity and dp/dz is constant along the pore.

Define $G \equiv \frac{1}{\mu}\frac{dp}{dz}$ (constant). Multiply by r and integrate:

$$\frac{d}{dr}\left(r\frac{du_z}{dr}\right) = Gr \quad \Longrightarrow \quad r\frac{du_z}{dr} = \frac{Gr^2}{2} + C_1.$$

Regularity at $r = 0$ requires $C_1 = 0$, so

$$\frac{du_z}{dr} = \frac{Gr}{2}.$$

Integrate once more:

$$u_z(r) = \frac{Gr^2}{4} + C_2.$$

No-slip at the wall $u_z(a) = 0$ gives

$$C_2 = -\frac{Ga^2}{4},$$

hence

$$u_z(r) = \frac{G}{4}\,(r^2 - a^2) = \frac{1}{4\mu}\frac{dp}{dz}\,(r^2 - a^2).$$

If the pressure drop across a pore of length h is $\Delta p = p_{\text{in}} - p_{\text{out}}$, then $dp/dz \approx -\Delta p/h$, and the axial profile may be written as the familiar Poiseuille form

$$\boxed{u_z(r) = \frac{\Delta p}{4\mu h}\,(a^2 - r^2).}$$

The volumetric flow through one pore is

$$\begin{aligned} Q_{\text{pore}} &= \int_0^{2\pi}\int_0^a u_z(r)\,r\,dr\,d\theta = 2\pi\frac{\Delta p}{4\mu h}\int_0^a (a^2 - r^2)r\,dr \\ &= 2\pi\frac{\Delta p}{4\mu h}\left[\frac{a^4}{4}\right] = \frac{\pi a^4}{8\mu}\,\frac{\Delta p}{h}. \end{aligned}$$

Thus the single-pore conductance scales with a^4 (Hagen–Poiseuille law).

b. Permeability for two pore sizes occupying specified surface fractions.

Let the membrane open-area fraction in pore class i be ϵ_i $(i = 1, 2)$, so $\epsilon_1 + \epsilon_2 = \epsilon$. The number of pores of class i per unit membrane area is

$$n_i = \frac{\epsilon_i}{\pi a_i^2}.$$

Each pore of class i has flow $Q_{\text{pore},i} = \dfrac{\pi a_i^4}{8\mu}\dfrac{\Delta p}{h}$, so the contribution to the superficial velocity (flow per unit membrane area) from class i is

$$U_i \equiv \frac{Q_i}{A} = n_i Q_{\text{pore},i} = \frac{\epsilon_i}{\pi a_i^2}\cdot\frac{\pi a_i^4}{8\mu}\frac{\Delta p}{h} = \frac{\epsilon_i a_i^2}{8\mu h}\,\Delta p.$$

Summing the two classes,

$$U \;=\; \frac{\Delta p}{8\mu h}\left(\epsilon_1 a_1^2 + \epsilon_2 a_2^2\right).$$

The problem statement defines permeability K via

$$\frac{Q}{A} \;=\; U \;=\; K\,\frac{\Delta p}{\mu}.$$

Comparing with the expression above yields

$$\boxed{K \;=\; \frac{1}{8h}\left(\epsilon_1 a_1^2 + \epsilon_2 a_2^2\right).}$$

This is the desired expression for K (note it scales like $\epsilon_i a_i^2/h$ and is independent of μ by construction of the chosen normalization).

Special case: equal surface-area occupation. If the two pore classes occupy equal fractions of the open area, i.e. $\epsilon_1 = \epsilon_2 = \epsilon/2$, then

$$\boxed{K = \frac{\epsilon\,(a_1^2 + a_2^2)}{16\,h}.}$$

c. Fraction of total flow through larger pores when $a_2 = 2a_1$.

With equal surface-area occupation ($\epsilon_1 = \epsilon_2$) the superficial flux contributed by class i is proportional to $\epsilon_i a_i^2$, so the fraction of total flow going through class 2 (the larger pores) is

$$\frac{U_2}{U} = \frac{\epsilon_2 a_2^2}{\epsilon_1 a_1^2 + \epsilon_2 a_2^2}.$$

With $\epsilon_1 = \epsilon_2$ and $a_2 = 2a_1$,

$$\frac{U_2}{U} = \frac{a_2^2}{a_1^2 + a_2^2} = \frac{(2a_1)^2}{a_1^2 + (2a_1)^2} = \frac{4}{1+4} = \frac{4}{5}.$$

Thus

$$\boxed{\text{80\% of the total flow passes through the larger pores.}}$$

Remarks and limitations.

- The derivation assumes independent parallel cylindrical conduits. Real membranes may be tortuous, connected, or have pore-size distributions that violate the simple area partitioning used here.
- Entrance/exit losses, inertia (nonzero Reynolds number), slip, and pore roughness are neglected; these effects can alter the prefactors and effective scaling.
- The strong dependence of single-pore flow on a^4 (but permeability on a^2 after area-weighting) shows large pores disproportionately increase overall membrane permeability.

3 The Clepsydra of the Ancient Greeks — Solution

Given. Conical bowl with internal cone angle 60 degrees so that the fluid volume and height are related by

$$V(h) = \frac{\pi}{9}\,h^3.$$

Small circular hole at the cone apex of radius R_0 (area $A_0 = \pi R_0^2$). Discharge coefficient λ (accounts for contraction and losses). Gravity g acts downward. Initial volume V_0 (equivalently initial height H_0 via $V_0 = \pi H_0^3/9$).

We treat the outflow as inertia-dominated (Torricelli type) with losses lumped into λ so that the volumetric outflow is

$$Q = \lambda\,A_0\,\sqrt{2gh}\,.$$

a. Differential equation for height (inertial scaling).

Mass (volume) conservation for the bowl:

$$\frac{dV}{dt} = -Q.$$

Differentiating $V = \dfrac{\pi}{9}h^3$ gives

$$\frac{dV}{dh} = \frac{\pi}{3}\,h^2, \qquad \text{so} \qquad \frac{dV}{dt} = \frac{\pi}{3}\,h^2\,\frac{dh}{dt}.$$

Set this equal to $-Q$:

$$\frac{\pi}{3}\,h^2\,\frac{dh}{dt} = -\lambda A_0\sqrt{2gh}.$$

Solve for dh/dt. Noting $\sqrt{h} = h^{1/2}$ and grouping constants, we obtain

$$\frac{dh}{dt} = -\underbrace{\left(\frac{3\lambda A_0\sqrt{2g}}{\pi}\right)}_{C_0}\,h^{-3/2}.$$

Thus the governing ODE is

$$\boxed{\frac{dh}{dt} = -C_0\,h^{-3/2}, \qquad C_0 = \frac{3\lambda A_0\sqrt{2g}}{\pi} = 3\lambda R_0^2\sqrt{2g}\,.}$$

b. Non-dimensionalization and characteristic time scale.

Let the initial height be H_0 (obtained from V_0 by $H_0 = (9V_0/\pi)^{1/3}$). Introduce dimensionless height and time

$$H^* = \frac{h}{H_0}, \qquad \tau = \frac{t}{t_c},$$

with an as-yet-unspecified characteristic time t_c. Substitute $h = H_0 H^*$ into the ODE:

$$\frac{H_0}{t_c}\frac{dH^*}{d\tau} = -C_0\,(H_0H^*)^{-3/2} = -C_0H_0^{-3/2}(H^*)^{-3/2}.$$

Choose the characteristic time to remove prefactors on the right:

$$t_c \equiv \frac{H_0^{5/2}}{C_0}.$$

With this choice the dimensionless ODE becomes

$$\frac{dH^*}{d\tau} = -(H^*)^{-3/2}.$$

Thus a convenient characteristic drainage time (scale) is

$$\boxed{t_c = \frac{H_0^{5/2}}{C_0} = \frac{H_0^{5/2}}{3\lambda R_0^2\sqrt{2g}}\,.}$$

c. Solve for the dimensionless drainage time t_d^*.

The dimensionless ODE

$$\frac{dH^*}{d\tau} = -(H^*)^{-3/2}$$

can be integrated analytically. Multiply both sides by $H^{*\,3/2}$ and use

$$\frac{d}{d\tau}\Big(H^{*\,5/2}\Big) = \frac{5}{2}H^{*\,3/2}\frac{dH^*}{d\tau}.$$

Hence

$$\frac{d}{d\tau}\Big(H^{*\,5/2}\Big) = -\frac{5}{2}.$$

Integrate from $\tau = 0$ (when $H^* = 1$) to arbitrary τ:

$$H^{*\,5/2}(\tau) = 1 - \frac{5}{2}\,\tau.$$

The drainage (empty) time corresponds to $H^* \to 0$, so set the left-hand side to zero and solve for τ_d:

$$0 = 1 - \frac{5}{2}\,\tau_d \quad\Longrightarrow\quad \tau_d = \frac{2}{5}.$$

Thus the *dimensionless* drainage time is

$$\boxed{\;t_d^* \;=\; \tau_d \;=\; \frac{2}{5}\,.\;}$$

The *actual* drainage time is therefore

$$\boxed{\;t_d \;=\; t_c\,t_d^* \;=\; \frac{2}{5}\,\frac{H_0^{5/2}}{3\lambda R_0^2\sqrt{2g}}.\;}$$

(Equivalently one can combine constants and write $t_d = (2/5)H_0^{5/2}/C_0$.)

d. Numerical design: find hole size for $V_0 = 1$ L and $t_d = 5$ min with $\lambda = 0.6$.

First compute the initial height H_0 from $V_0 = 1.0\text{ L} = 1.0 \times 10^{-3}\text{ m}^3$:

$$H_0 = \left(\frac{9V_0}{\pi}\right)^{1/3}.$$

Numerically,

$$H_0 \approx \left(\frac{9(1.0\times 10^{-3})}{\pi}\right)^{1/3} \approx 0.1420\text{ m} \quad \text{(about 14.2 cm)}.$$

Rearrange the expression for t_d to solve for R_0:

$$t_d = \frac{2}{5}\,\frac{H_0^{5/2}}{3\lambda R_0^2\sqrt{2g}} \quad\Longrightarrow\quad R_0^2 = \frac{2}{5}\,\frac{H_0^{5/2}}{3\lambda\sqrt{2g}\;t_d}.$$

Using $t_d = 5$ min $= 300$ s, $\lambda = 0.6$, and $g = 9.80665$ m/s^2,

$$H_0 \approx 0.1420248 \text{ m},$$
$$R_0 \approx 1.128 \times 10^{-3} \text{ m} \;=\; 1.128 \text{ mm}.$$

So the hole radius must be about 1.13 mm (equivalently diameter about 2.26 mm) to achieve a 5-minute drainage time with the given discharge coefficient.

$$\boxed{R_0 \approx 1.13 \times 10^{-3} \text{ m}, \qquad D_0 \approx 2.26 \times 10^{-3} \text{ m}.}$$

e. Validity of the inertial approximation (quantitative check).

The inertial (Torricelli) form $Q = \lambda A_0 \sqrt{2gh}$ is derived assuming inertia in the orifice dominates viscous stresses there; a common check is the Reynolds number based on the jet/orifice diameter and the jet velocity. Use the initial conditions (largest velocity and therefore largest Reynolds):

Initial jet velocity (approximate) at $h = H_0$:

$$U_0 = \lambda\sqrt{2gH_0}.$$

Numerically (with $\lambda = 0.6$ and $H_0 \approx 0.1420\,\text{m}$),

$$U_0 \approx 1.00 \text{ m/s}.$$

Characteristic length for the jet/orifice is the diameter $D_0 = 2R_0 \approx 2.26 \times 10^{-3}$ m. For water (take $\rho \approx 1000$ kg/m^3 and $\mu \approx 1 \times 10^{-3}$ Pa $\cdot$ s) the orifice Reynolds number is

$$\text{Re}_{\text{orifice}} = \frac{\rho U_0 D_0}{\mu} \approx \frac{1000 \times 1.00 \times 2.26 \times 10^{-3}}{1 \times 10^{-3}} \approx 2.26 \times 10^3.$$

This is well into the inertia-dominated regime (Re $\gg 1$), so viscous stresses in the orifice are small compared with inertia and the Torricelli-type formula with a discharge coefficient is appropriate. The chosen discharge coefficient $\lambda = 0.6$ reasonably accounts for contraction/energy losses and brings the simple model into good quantitative agreement with realistic outflow.

Additional remarks.

- The discharge coefficient λ is important; reducing λ (more losses) increases the required hole size for a given time. Typical orifice values for thin-walled sharp-edged holes are $\lambda \approx 0.6$–0.8, while long thin pipes would have much smaller effective λ.
- The quasi-steady assumption (that the instantaneous Torricelli formula applies while $h(t)$ slowly changes) is justified when the time scale for adjustment of the jet is much shorter than the drainage time; here $U_0/H_0 \sim 1/0.14 \sim 7$ s^{-1} so the jet adjusts in fractions of a second whereas the drainage time is minutes.

- If a more accurate prediction is required one could account for unsteady Bernoulli corrections, viscous head losses in the hole, or energy loss in the free jet; these effects are partially captured by a measured λ.