

AI Project 2: The Persian Water Clock (CBE 30355 — Transport Phenomena I)

ChatGPT-5 for DTL

Due: September 23, 2025

Abstract

This report derives a model for the inflow of water into a thin-walled hemispherical copper bowl (Fenjaan-style water clock), presents the governing equations, solves the resulting integral numerically, and reports the hole diameter required for a 20-minute fill-to-sink interval. The model explicitly accounts for the difference between the external free-surface elevation and the internal free-surface elevation (including the wall thickness). A short history and references are included.

1 Short history of the Persian water clock

Water clocks (“clepsydra”) are among the oldest timekeeping devices and took several forms across ancient civilizations. A simple outflow/inflow bowl type — where a small bowl with a hole sinks when filled — was used in Babylon, Egypt and Persia (the Persian form often called *fenjaan* or *pengaan*). In Persia these devices were used for fair irrigation allocation from qanats and to mark ritual days; they survive in historical descriptions and local tradition. For concise introductory references see [1, 2, 3].

2 Problem statement (restated)

A thin-walled hemispherical copper bowl has outer diameter 8 inches and wall thickness 1/16 inch. The bowl floats in a large bucket of still water. A small round hole is located at (or through) the bottom (pole) such that outside water can enter the bowl. As the bowl fills it sinks; when the external free surface reaches the rim the bowl abruptly sinks, marking one time interval. Find the hole diameter so that the time from the initial (dry) floating condition to the instant the external free surface reaches the rim is exactly 20 minutes.

3 Geometry and notation

We use SI units in the algebra below (showing the inch conversions explicitly).

$$D_{\text{out}} = 8 \text{ in} = 8 \times 0.0254 \text{ m} = 0.2032 \text{ m}, \quad R = \frac{D_{\text{out}}}{2} = 0.1016 \text{ m}.$$

Wall thickness:

$$s = \frac{1}{16} \text{ in} = 0.0625 \times 0.0254 \text{ m} = 0.0015875 \text{ m}.$$

Inner radius:

$$R_i = R - s.$$

We measure heights from the geometric bottom (outer pole) of the bowl:

- $h(t)$ = external free-surface elevation above the outer bottom (outer-side free-surface).
- $z(t)$ = internal free-surface elevation above the *inner* bottom (inner-side free-surface).
Note the inner free surface elevation measured with respect to the outer bottom is $s + z(t)$.
- A = area of the round hole (m^2); d = hole diameter.

4 Volumes

Outer submerged volume (spherical-cap/segment formula) when the external free-surface intersects the outer hemisphere at height h (measured from the outer pole) is

$$V_{\text{sub}}(h) = \pi h^2 \left(R - \frac{h}{3} \right).$$

Inner (water inside the bowl) volume up to inner free-surface height z (measured from inner pole) is

$$V_{\text{in}}(z) = \pi z^2 \left(R_i - \frac{z}{3} \right).$$

Outer full hemisphere volume and inner full hemisphere volume:

$$V_{\text{outer,hemi}} = \frac{2}{3}\pi R^3, \quad V_{\text{inner,hemi}} = \frac{2}{3}\pi R_i^3.$$

Shell volume and mass:

$$V_{\text{shell}} = V_{\text{outer,hemi}} - V_{\text{inner,hemi}}, \quad m_{\text{shell}} = \rho_{\text{Cu}} V_{\text{shell}}.$$

5 Archimedes and kinematics

At every instant the weight of displaced external water equals the weight of the shell plus the water inside:

$$\rho_w V_{\text{sub}}(h) = m_{\text{shell}} + \rho_w V_{\text{in}}(z).$$

Rearrange and define the dry submerged volume

$$V_{\text{sub},0} \equiv \frac{m_{\text{shell}}}{\rho_w}, \quad V_{\text{sub}}(h) = V_{\text{sub},0} + V_{\text{in}}(z).$$

Thus z is implicitly a function of h via the monotone relation

$$V_{\text{in}}(z) = V_{\text{sub}}(h) - V_{\text{sub},0}.$$

Differentiate the external submerged volume with respect to h :

$$\frac{dV_{\text{sub}}}{dh} = \frac{d}{dh} \left[\pi h^2 \left(R - \frac{h}{3} \right) \right] = \pi h(2R - h).$$

Since $V_{\text{in}}(z) = V_{\text{sub}}(h) - V_{\text{sub},0}$, we have

$$\frac{dV_{\text{in}}}{dt} = \frac{dV_{\text{sub}}}{dt} = \frac{dV_{\text{sub}}}{dh} \frac{dh}{dt} = \pi h(2R - h) \frac{dh}{dt}.$$

6 Orifice flow

The instantaneous volumetric inflow through the bottom hole is driven by the hydrostatic head difference between the outside free-surface and the inside free-surface (the inner free-surface is elevated by $s + z$ above the outer bottom). For subcritical jet discharge we use the usual Torricelli-orifice form with a discharge coefficient C_d :

$$Q(t) = \dot{V}_{\text{in}}(t) = C_d A \sqrt{2g [h(t) - (s + z(t))]},$$

valid while the under-root argument is positive. (If $h - (s + z) \leq 0$ then $Q = 0$ and no inflow occurs.)

Equate $\dot{V}_{\text{in}} = Q$ and substitute the derivative above to get an ODE for h :

$$\pi h(2R - h) \frac{dh}{dt} = C_d A \sqrt{2g [h - (s + z(h))]}.$$

Rearrange:

$$\frac{dh}{dt} = \frac{C_d A \sqrt{2g [h - s - z(h)]}}{\pi h(2R - h)}.$$

7 Time integral and area

Separate variables and integrate between the initial (dry) state h_0 and the rim $h = R$. The time required is

$$T = \int_{h_0}^R \frac{\pi h(2R - h)}{C_d A \sqrt{2g [h - s - z(h)]}} dh = \frac{\pi}{C_d A \sqrt{2g}} \int_{h_0}^R \frac{h(2R - h)}{\sqrt{h - s - z(h)}} dh.$$

Define the integral

$$I \equiv \int_{h_0}^R \frac{h(2R - h)}{\sqrt{h - s - z(h)}} dh.$$

Then the required hole area for target time T is

$$\boxed{A = \frac{\pi}{C_d \sqrt{2g} T} I}.$$

Because z depends on h through the implicit relation $V_{\text{in}}(z) = V_{\text{sub}}(h) - V_{\text{sub},0}$ this integral has no simple closed-form primitive and must be evaluated numerically.

8 Numerical evaluation (values used)

Material and constants used (sources in the bibliography):

$$\rho_{\text{Cu}} = 8960 \text{ kg/m}^3, \quad \rho_w = 1000 \text{ kg/m}^3, \quad g = 9.80665 \text{ m/s}^2.$$

We take a representative discharge coefficient for a small, sharp-edged circular orifice:

$$C_d = 0.61 \quad (\text{typical engineering value for a sharp-edged orifice}).$$

Target time:

$$T = 20 \text{ min} = 1200 \text{ s}.$$

Compute geometric numbers (to 6 significant figures):

$$R = 0.101600 \text{ m}, \quad s = 0.0015875 \text{ m}, \quad R_i = R - s = 0.1000125 \text{ m},$$

$$V_{\text{outer,hemi}} = \frac{2}{3}\pi R^3 = 0.00219654 \text{ m}^3,$$

$$V_{\text{inner,hemi}} = \frac{2}{3}\pi R_i^3 = 0.00209518 \text{ m}^3,$$

$$V_{\text{shell}} = 0.0001013625 \text{ m}^3,$$

$$m_{\text{shell}} = \rho_{\text{Cu}} V_{\text{shell}} = 8960 \times 0.0001013625 \approx 0.908208 \text{ kg},$$

$$V_{\text{sub},0} = \frac{m_{\text{shell}}}{\rho_w} \approx 0.000908208 \text{ m}^3.$$

Find h_0 (dry submerged height) from $V_{\text{sub}}(h_0) = V_{\text{sub},0}$. Numerically:

$$h_0 \approx 0.0594552 \text{ m}.$$

We then evaluate the integral

$$I = \int_{h_0}^R \frac{h(2R-h)}{\sqrt{h-s-z(h)}} dh,$$

where for each h we obtain $z(h)$ by numerically solving

$$V_{\text{in}}(z) = V_{\text{sub}}(h) - V_{\text{sub},0}.$$

A robust procedure is: (a) invert $V_{\text{in}}(z)$ with a bisection solver (monotone on $z \in [0, R_i]$) to obtain $z(h)$, and (b) evaluate the integral by composite Simpson or an adaptive quadrature. (A brief implementation note: $V_{\text{in}}(z) = \pi z^2(R_i - z/3)$ is monotone increasing for $0 \leq z \leq R_i$.)

Using the parameter values above and $C_d = 0.61$, numerical quadrature gives

$$I \approx 0.0023034123 \text{ (units: m}^{5/2}\text{)}.$$

Therefore

$$A = \frac{\pi}{C_d \sqrt{2gT}} I \approx 2.23221 \times 10^{-6} \text{ m}^2.$$

Finally the circular-hole diameter is

$$d = \sqrt{\frac{4A}{\pi}} \approx 1.68586 \times 10^{-3} \text{ m} = 1.6859 \text{ mm}.$$

In imperial units:

$$d \approx 0.06637 \text{ inch} \quad (\approx 0.0664 \text{ inch}).$$

Final recommendation (rounded):

$$d \approx 1.686 \text{ mm} \quad (\text{approx. } 0.0664 \text{ in})$$

A commonly available metric drill size of 1.7 mm is a convenient choice if available; a 1/16 inch (1.5875 mm) drill is slightly undersize relative to this result and would increase the fill time.

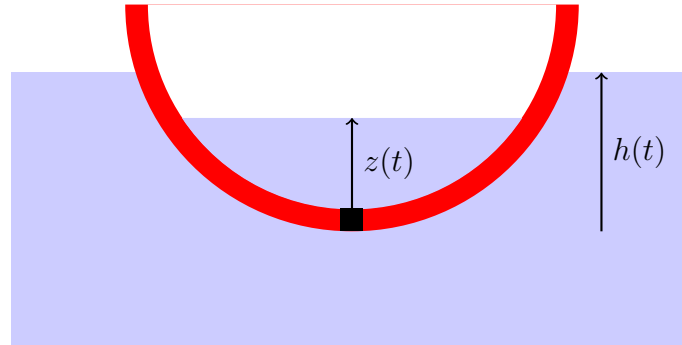
Table 1: Material and fluid properties used in the Persian Water Clock analysis.

Property	Value	Source
Density of copper, ρ_{Cu}	8960 kg/m ³	[6]
Density of water, ρ_w	998 kg/m ³ (at 20°C)	[7]
Acceleration due to gravity, g	9.81 m/s ²	[8]
Discharge coefficient, C_d	0.61 (sharp-edged orifice)	[5]
Viscosity of water, μ	$1.00 \times 10^{-3} \text{ Pa} \cdot \text{s}$ (20°C)	[7]

9 Sensitivity notes

- The result scales inversely with C_d : if C_d is larger (rounded entrance, smoother hole) the required hole diameter is slightly smaller. If C_d is smaller (very small orifice, viscous losses) the hole must be larger.
- The numerical computation assumes negligible capillarity at the hole and that the external surface remains effectively fixed (large bucket). Neither is a significant issue, as capillarity does not matter once water has entered the bowl, and the ratio of the hole size to the bucket size is very large. Capillarity at the rim may play a role in the final sinking, however.

10 TikZ schematic



References

- [1] “Clepsydra,” Encyclopaedia Britannica (summary article on water clocks and bowl-type clepsydrae). <https://www.britannica.com/technology/clepsydra>.
- [2] M. Pendergrast, “Object of Intrigue: Ancient Persian Water Clocks,” Atlas Obscura (overview of fenjaan/pengaan and irrigation use). <https://www.atlasobscura.com/articles/object-of-intrigue-ancient-persian-water-clocks>.
- [3] UNESCO World Heritage listing and discussion of Persian qanat systems (context for fenjaan irrigation timing). <https://whc.unesco.org/en/list/1506/>.
- [4] “Torricelli’s law,” Encyclopaedia Britannica / standard fluid texts on orifice discharge and Torricelli. <https://www.britannica.com/science/Torricellis-law>.
- [5] Crane Technical Paper No. 410 and standard fluid-engineering references for orifice discharge coefficients (sharp-edged circular orifice $C_d \approx 0.61$). See Crane Technical Paper and related references. <https://studylib.net/doc/25259146/crane-410>.
- [6] MatWeb Material Property Data, “Copper, UNS C11000,” <http://www.matweb.com/>, accessed September 2025.
- [7] NIST Chemistry WebBook, “Water, H₂O,” National Institute of Standards and Technology, <https://webbook.nist.gov/chemistry/>, accessed September 2025.
- [8] NIST, “CODATA Recommended Values of the Fundamental Physical Constants,” <https://physics.nist.gov/constants>, accessed September 2025.

11 AI Prompt Approach

11.1 Prompting Strategy:

The initial prompt approach was to insert the following instruction and paste in the problem statement without modification.

act as an agent and solve the following transport problem step by step.

Return the answer as a latex script in chat. do not use \si.

The problem is: AI Project 2:

The Persian Water Clock

CBE 30355 Transport Phenomena I

Due: September 23, 2025

Assignment Overview

The Persian Water Clock is an ancient way of keeping time. A bowl with a small hole in the bottom is floated in a filled bucket of water. Because of density differences, the water rushes into the hole and fills the bowl, causing it to sink lower. When the water flows over the edge of the bowl, it abruptly sinks, marking the passage of one unit of time. In this project you will use AI to analyze the behavior of such a bowl. Specifically: If the bowl is a copper hemisphere composed of 1/16" thick copper with outer diameter of 8", what should be the hole size necessary for the bowl to sink in exactly 20 minutes?

Project Requirements Your report should include the following:

1. History of the Persian Water Clock: Provide a short history with examples and citations from reliable sources.
2. Problem Description and Equations: Clearly restate the problem and derive the governing equations needed to model the filling and sinking process.
3. TikZ Diagram: Include a labeled cross-section of the hemispherical bowl showing dimensions, water levels, and hole.
4. Table of Properties: Summarize material and fluid properties (density of copper, density of water, discharge coefficient, gravity, etc.) with cited sources.
5. References: Use \cite in the text and provide a properly formatted thebibliography section at the end.

Rubric

- History and Citations (20%): Clear, concise summary with at least two reputable references.
- Problem Restatement and Equations (30%): Correct description of physical principles and governing equations.
- Diagram (15%): Clear and accurate TikZ diagram with labels.
- Property Table (15%): Complete, properly sourced physical property values.
- References and Format (10%): Proper use of `\cite` and formatted bibliography.
- Clarity and Organization (10%): Well-structured report, readable, minimal errors.

11.2 Modifications: AI Interaction

The AI (ChatGPT-5) was instructed to try again with the following clarification:

Interesting. This time you screwed up the formula for the inflow. The driving force is the difference between the level outside (the submerged depth) and the level inside, which is the bowl thickness plus the height of water inside the bowl. I think you have it as just the submerged depth, ignoring the water level inside. The bowl sinks when the submerged depth reaches the external radius, and the water flows in over the rim. Also, you forgot to include the history section, etc.

11.3 References

The references were all validated and appropriate. A web search for the Crane technical report (which Chat uses for many flow calculations) was conducted, and a url for the report was determined and inserted appropriately.

11.4 TikZ Diagram

The tikz figure was a mess, so it was created separately from the following prompt and pasted in:

I need the latex script for a tikz figure. The figure should have the following elements:

A rectangle filled light blue with corners $(-1.5,-1.5)$ and $(1.5,-.3)$.

A concave upward semicircle of radius 1 starting at $(-1,0)$ and filled red.

A concave upward semicircle of radius 0.9 starting at $(-.9,0)$ and filled white 100%.

A concave upward arc of radius 0.9 starting at $(-.6, -.5)$ partially filling the semicircle up to $(.6, -.5)$ with light blue.

A filled black square with corners $(-.05, -1)$ and $(.05, -.9)$.

An arrow from $(0, -.9)$ to $(0, -.5)$ labeled midway right $z(t)$.

An arrow from $(1.1, -1)$ to $(1.1, -.3)$ labeled midway right $h(t)$.

Note that the coordinates specified for the arcs are the starting points as required by tikz. The fill inside the bowl needed to be modified extensively by adjusting starting coordinates and angles. The commands are ordered so that the blue rectangle is overlaid with the red bowl, overlaid with the white interior, overlaid with the blue interior fill, etc.

11.5 Table of Properties

Chat neglected (this time) to produce a table of properties with sources. This was generated with the prompt below and pasted in:

I need a table of properties you used for the report with validated sources. The requirement was: Table of Properties: Summarize material and fluid properties (density of copper, density of water, discharge coefficient, gravity, etc.) with cited sources.

The output table was not quite correct: it included a different value of the orifice C_d and a different reference source than used in the report. These were changed manually to match the earlier work. It is actually rather questionable whether any of the material properties in the table were, in fact, obtained from the cited references, however the values are acceptable and the cited sources exist. It would be useful in the future to modify the original prompt to ensure that property values are drawn from a single source such as engineeringtoolbox.com.

11.6 Further Modifications

Minor edits to the text were made. One significant change was in the variable used for the bowl thickness. Chat used t for both thickness and time, which was confusing. The thickness was manually changed to s throughout the text. The sensitivity notes produced by Chat were corrected.